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Momentum/heat transfer analogy for power-law fluids during turbulent boundary layer flow with mild pressure gradients

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Abstract—There have been many attempts in the literature to develop analogies for momentum, heat and mass transfer to power-law fluids. However, none consider the presence of a pressure gradient when formulating the analogies. The present work attempts to develop a momentum/heat transfer analogy under the influence of a mild pressure gradient for non-Newtonian power-law fluids using the Nakayama *et al.* (*AIAA J.* **22,** 841–844 (1984)) solution methodology for Newtonian fluids.

INTRODUCTION

It is well-known that the heat transfer rate can be reasonably well estimated without actually solving the energy equation through the use of momentum/heat transfer analogies. For Newtonian fluids, the number of alternative approaches for establishing the momentum and heat transfer during turbulent flow have been discussed by Nakayama *et al.* [1] and they have themselves derived the momentum and heat transfer analogy for external turbulent boundary layer flow under the influence of mild pressure gradients.

In the case of non-Newtonian power-law fluids, there have been attempts to establish momentum/heat transfer analogies by Metzner and Friend [2], Skelland [3], Petersen and Christiansen [4], Krantz and Wasan [5], Sandall *et al.* [6], Smith and Edwards [7], Kawase and Ulbrecht [8], Irvine and Karni [9] and Wangskarn and Ghorashi [10].

Skelland [3] and Irvine and Karni [9] have provided heat transfer analogies by using a Blasius type relationship between friction factor and Reynolds number for the external turbulent flow past the flat plate whereas the rest have analyzed the internal flow through smooth circular pipes. Metzner and Friend [2] calculated the Stanton number as a function of the friction factor and Prandtl number, applying Riechardt's general formulation for the analogy between heat and momentum transfer in turbulent pipe flow. Their correlation gave fairly good predictions for purely viscous non-Newtonian fluids. Petersen and Christiansen [4] extended the Metzner–Friend correlation to non-isothermal and transitional flow and claimed an improvement in the heat transfer prediction by the use of a modified Prandtl number. Krantz and Wasan [5] presented a correlation for heat, mass and momentum transfer in the fully developed turbulent flow of power-law fluids in circular tubes which has the same form as the Metzner–Friend correlation but differs from it in terms of the use of the continuous eddy viscosity distribution. Sandall *et al.* [6] reprocessed the data generated by Raniere [11], Haines [12], Friend [13] and Farmer [14] and came up with a new correlation for Stanton number. Smith and Edwards [7] extended the eddy viscosity expression for Newtonian pipe flow to non-Newtonian flow by using the apparent viscosity at the wall. Kawase and Ulbrecht [8] proposed a new theoretical expression using Levich's three-zone model for predicting turbulent heat and mass transport in inelastic non-Newtonian liquids. Wangskarn and Ghorashi [10] proposed a model for heat transfer to non-Newtonian power-law fluids flowing through heated horizontal pipes which was shown by them to be applicable to a wide range of flow behavior index.

Though there are a number of correlations available as stated above, none of them have considered the presence of pressure gradients during the turbulent boundary layer flow. In the present paper, the Nakayama *et al.* [1] solution method for Newtonian fluids is extended to non-Newtonian power-law fluids in order to establish the momentum/heat transfer analogy in the presence of mild pressure gradients.

ANALYSIS

The total shear stress at any point in a turbulent fluid consists of a viscous shear component and a turbulent shear component given as

$$\tau = \tau_{\text{viscous}} + \tau_{\text{turbulent}}. \tag{1}$$

For non-Newtonian inelastic fluids, it is assumed that the flow behavior is well described by the power-law model and hence the total shearing stress can be written in line with the well-known Prandtl mixing length theory used earlier by Clapp [15] as follows:

NOMENCLATURE

A	coefficient in equation (7) and defined by equation (8a)
B	coefficient in equation (7) and defined by equation (8b)
C	coefficient in equation (7) and defined by equation (8c)
C_{fx}	local skin friction coefficient defined by equation (14b)
C_p	specific heat per unit mass
d	pipe diameter
f	fanning friction factor defined in equation (18)
I	function defined in equation (23b)
k	thermal conductivity of the fluid
K	consistency index of the power-law fluid
m	coefficient in the summation series given by equation (5a)
$\bar{m}$	function defined by equation (24c)
n	pseudoplasticity index of the power-law fluid
p	pressure in equation (10a)
P	'P-function' defined by equation (13b)
Pr_N	Prandlt number for Newtonian fluid in equation (13b)
Pr_w	Prandlt number evaluated using viscosity of fluid at wall shearing stress and defined by equation (13c)
Pr_x	Prandtl number for power-law fluids in external flows and defined by equation (27a)
q	heat flux in equation (10b)
q_w	heat flux at the wall
Re	Reynolds number for power-law fluids in pipe flow in equation (17) and defined as $\rho V^{2-n}d^n/\gamma_1$
Re_x	Reynolds number for power-law fluids in external flows and defined by equation (23c)
s_p	drag coefficient in equation (13a)
St	Stanton number defined by equation (30a)
St_x	Stanton number defined by equation (14c)
T	temperature
T_b	temperature of the bulk of the fluid
T_e	temperature at the edge of the boundary layer
T_w	temperature at the wall
u	streamwise velocity component
u_e	velocity at the edge of the boundary layer
u_m	maximum velocity in pipe flow
u^+	dimensionless velocity defined by equation (8d)
V	average velocity in pipe flow
x, y	boundary layer coordinates
y_s	distance from the wall defined by equation (9b)
y^+	dimensionless distance defined by equation (8e).

Greek symbols

α, β	dimensionless functions of n appearing in equation (17a)
β_1	pressure gradient function defined by equation (5b)
γ_1	function of n defined by equation (20)
δ	viscous (velocity) boundary layer thickness
δ_t	thermal boundary layer thickness
κ	proportionality constant between mixing length and distance y defined in equation (9a)
μ	viscosity of the fluid
ρ	density of the fluid
τ	total shear stress defined in equation (1)
τ_w	wall shear stress
τ_{viscous}	viscous shear component defined in equation (2)
$\tau_{\text{turbulent}}$	turbulent shear component defined in equation (2)
Ω	coefficient defined in equation (19).

$$\tau = K\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy} + \rho(\kappa y)^2\left|\frac{du}{dy}\right|\frac{du}{dy} \tag{2}$$

where u and τ are the mean velocity in the streamwise direction x and the local shear stress at the normal distance y away from the wall. K is the consistency index and n the power-law index describing the rheological behavior of the fluid. The density is denoted as ρ and the proportionality constant between mixing length and distance y is denoted as κ. For Newtonian fluids, κ is the von Karman constant and for power-law fluids this would be derived later from the known velocity profile for power-law fluids.

It is now assumed that the turbulent shear dominates the flow situation and that the viscous shear can be neglected in comparison to its magnitude when describing the total shear. Thus, we have

$$\tau = \rho\left(\kappa y\frac{du}{dy}\right)^2. \tag{3}$$

The shear stress near the wall is usually known to vary as follows:

$$\tau = \tau_w + \left(\frac{d\tau}{dy}\right)_w y \tag{4}$$

where the subscript w refers to the wall. Equations (3) and (4) can be combined to give

$$\frac{du}{dy} = \frac{(\tau/\rho)^{1/2}}{\kappa y} = [1+\beta_1(y/\delta)]^{1/2}\frac{(\tau_w/\rho)^{1/2}}{\kappa y}$$
$$= \left[1+\sum_1^{\infty}\binom{1/2}{m}(\beta_1 y/\delta)^m\right]^{1/2}\frac{(\tau_w/\rho)^{1/2}}{\kappa y} \quad (5a)$$

where

$$\beta_1 = \frac{\delta}{\tau_w}\left(\frac{d\tau}{dy}\right)_w \quad (5b)$$

$$\binom{1/2}{m} = 1/2(1/2-1)(1/2-2)\cdots(1/2-m+1)/m! \quad (5c)$$

where δ is the viscous (velocity) boundary layer thickness. The stress gradient is assumed to be mild enough such that $|\beta_1 y/\delta| < 1$. Equation (5a) may be readily integrated to yield

$$\frac{u}{(\tau_w/\rho)^{1/2}} = \frac{1}{\kappa}\ln(y/y_s)+\frac{1}{\kappa}\sum_1^{\infty}\binom{1/2}{m}\frac{\beta_1^m}{m}\left(\frac{y^m-y_s^m}{\delta^m}\right). \quad (6)$$

Power-law fluid velocity profiles in turbulent pipe-flow have been discussed in detail in ref. [16]. An expression for the velocity profile in ref. [17] is adapted to the boundary layer flow situation under consideration to give the following:

$$u^+ = A\ln y^{+(1/n)}+(AC+B) \quad (7)$$

where

$$A = 2.46n^{0.25} \quad (8a)$$
$$B = -0.4\sqrt{2}/n^{1.2} \quad (8b)$$
$$C = (0.1944-0.1313/n+0.3876/n^2-0.0109/n^3)$$
$$\times\exp(-4.961n^2)+1.3676/n+\ln 2^{(2+n)/2n} \quad (8c)$$
$$u^+ = u/(\tau_w/\rho)^{1/2} \quad (8d)$$
$$y^+ = y^n(\tau_w/\rho)^{(2-n)/2}\rho/K. \quad (8e)$$

The comparison of equation (6) with equation (7) for $\beta_1 = 0$ implies that

$$\kappa = 1/A = 0.4065/n^{0.25} \quad (9a)$$
$$y_s^n(\tau_w/\rho)^{(2-n)/2}\rho/K = \exp[-n(AC+B)/A]. \quad (9b)$$

Expression (9b) is evaluated for different values of n as given in Table 1. It can be seen that the value of 0.113 for $n = 1$ is close to the approximate value of 0.1 obtained by Nakayama *et al.* [1] for Newtonian fluids.

Table 1

n	α	β	$\exp[-n(AC+B)/A]$
1.0	0.0790	0.250	0.1130
0.9	0.0770	0.255	0.1177
0.8	0.0760	0.263	0.1219
0.7	0.0752	0.270	0.1249
0.6	0.0740	0.281	0.1249
0.5	0.0723	0.290	0.1198
0.4	0.0710	0.307	0.1070
0.3	0.0683	0.325	0.0844
0.2	0.0646	0.349	0.0526

Since the advection terms become small near the wall, the momentum and energy equations reduce to

$$\frac{d\tau}{dy} = \frac{dp}{dx} \quad (10a)$$

$$\frac{dq}{dy} = 0 \quad (10b)$$

where the pressure and heat flux are denoted by p and q. Equations (10) imply that the temperature profile near the wall may become fairly insensitive to the pressure gradient, while the velocity profile there must correspond to the pressure gradient according to equations (10a) and (5b)

$$\beta_1 = \frac{\delta}{\tau_w}\frac{dp}{dx} = -\frac{\rho\delta}{\tau_w}u_e\frac{du_e}{dx}. \quad (11)$$

The preceding observation on the energy equation indicates that the temperature law of the wall for zero pressure gradient given below may well be valid even for the case of mild pressure gradients

$$\frac{\rho C_p(\tau_w/\rho)^{1/2}}{q_w}(T_w-T) = A\ln(y/y_s)+P \quad (12)$$

where T and C_p are the temperature and specific heat, respectively, and P the Jayatillaka [18] 'P-function' that accounts for the enhanced resistance to heat transfer offered by the viscous sublayer as a function of laminar Prandtl number Pr. For Newtonian fluids, Jayatillaka [18] assumed a velocity profile of a form similar to equation (7) and derived an expression relating the extra resistance function $\sigma_0 P$, the drag coefficient s_p and the Stanton number St as follows:

$$\sigma_0 P = \frac{s_p^{1/2}}{St} - \frac{\sigma_0}{s_p^{1/2}}(1+1.25A^2 s_p) \quad (13a)$$

where σ_0 is the total Prandtl number in the fully turbulent region of the fluid, P the 'P-function', and the drag coefficient s_p is defined as $\tau_w/\rho V^2$ with V being the average velocity. Using a large amount of experimental values from the literature on Newtonian fluids, Jayatillaka [18] drew out the following simple form for the P-function which predicted the extra resistance to heat transfer rather accurately:

$$P = 9.24(Pr_N^{3/4}-1). \quad (13b)$$

For power-law fluids, the same procedure could be followed for the derivation of the 'P-function' as Jayatillaka [18]. In fact, using equation (7), an expression identical to equation (13a) can be easily obtained. However, in order to get an expression like equation (13b), a lot of accurate flow and heat transfer data on power-law fluids is required. There is certainly no dearth of such heat transfer data in the literature. Nevertheless, as a first approximation, it is assumed that equation (13b) holds for power-law fluids when the Prandtl number is defined appropriately in the form of Pr_w as follows:

$$Pr_w = \frac{\mu_w C_p}{k} \quad (13c)$$

where μ_w is the viscosity of the fluid evaluated at wall shearing stress. This definition of Prandtl number follows the one presented by Metzner and Friend [2].

In equations (12) and (13b), the turbulent Prandtl number is assumed to be unity. After evaluating equations (6) and (12) at the viscous ($y = \delta$) and the thermal ($y = \delta_t$) boundary layer edges, respectively, the subtraction of equation (12) from equation (6) leaves the following:

$$\left(\frac{2}{C_{fx}}\right)^{1/2} - \frac{(C_{fx}/2)^{1/2}}{St_x} = A \ln(\delta/\delta_t) - P + A \sum_1^{\infty} \binom{1/2}{m} \frac{\beta_1^m}{m}\left(1 - \frac{y_s^m}{\delta^m}\right) \quad (14a)$$

where the skin friction coefficient is

$$C_{fx} = 2\tau_w/\rho u_e^2 \quad (14b)$$

and the Stanton number

$$St_x = q_w/\rho C_p u_e(T_w - T_e). \quad (14c)$$

Subscript e refers to the corresponding boundary-layer edge $y = \delta$ or δ_t. Due to equation (9b), (y_s/δ) in the last term of the right-hand side of equation (14a) may be dropped. Moreover, the logarithmic term in equation (14a) can be neglected since $\ln(\delta/\delta_t) \sim 0$ for $Pr_w \sim 1$ and $\ln(\delta/\delta_t) \ll P/A$ for $Pr_w \gg 1$. Thus, equation (14a) reduces to the following compact form for the momentum/heat transfer analogy of present concern:

$$\frac{2St_x}{C_{fx}} = \left\{1 + \left(\frac{C_{fx}}{2}\right)^{1/2}\left[P - A\sum_1^{\infty}\binom{1/2}{m}\frac{\beta_1^m}{m}\right]\right\}^{-1}. \quad (15)$$

RESULTS AND DISCUSSION

A simple integral approach is now followed in order to get estimates of C_{fx} and β_1 so that the validity of equation (15) may be substantiated. A usual control volume analysis leads to the momentum balance relation given below:

$$\frac{d}{dx}\int_0^{\delta}(u_e u - u^2)\,dy + \frac{du_e}{dx}\int_0^{\delta}(u_e - u)\,dy = \frac{\tau_w}{\rho}. \quad (16)$$

For power-law fluids, Dodge and Metzner [19] have provided a Blasius-type of approximate equation for the friction factor in terms of the generalized Reynolds number relationship given as

$$f = \frac{\alpha}{Re^{\beta}} \quad 5\times 10^3 \leqslant Re \leqslant 10^5 \quad (17a)$$

where α and β are functions of n for the case of power-law fluids, and their values for varying n are presented in Table 1. Re is the generalized Reynolds number defined as follows:

$$Re = \frac{\rho V^{2-n} d^n}{\gamma_1}. \quad (17b)$$

Following the procedure of Skelland [20], a suitable expression for the local surface shear stress can be obtained from equation (17a) as follows:

$$C_{fx} = 2\tau_w/\rho u_e^2 = 2\Omega(\gamma_1/\rho u_e^{2-n}\delta^n)^{\beta} \quad (18)$$

where

$$\Omega = \alpha(0.817)^{2-\beta(2-n)}/2^{\beta n+1} \quad (19)$$

and

$$\gamma_1 = 8^{n-1}K[(3n+1)/4n]^n. \quad (20)$$

Note that for the Newtonian case: $n = 1$, $\alpha = 0.791$, $\beta = 0.25$, $\Omega = 0.02332$

$$C_{fx} = 0.04664(\mu/\rho u_e \delta)^{1/4}. \quad (21)$$

Equation (18) corresponds to the following velocity model for power-law fluids:

$$u/u_e = (y/\delta)^{\beta n/[2-\beta(2-n)]}. \quad (22)$$

Upon substitution of equations (18) and (22), equation (16) can be easily solved for δ to give

$$\frac{\delta}{x} Re_x^{\beta/(1+\beta n)} = \left\{\frac{2\Omega[2-\beta(2-3n)][1-\beta(1-n)][1+\beta n]}{\beta n[2-\beta(2-n)]}\right\}^{1/(1+\beta n)} I^{1/(1+\beta n)} \quad (23a)$$

where

$$I = \frac{\int_0^x u_e^{\{[3-2\beta(1-n)][2-\beta(2-3n)]\}/[2-\beta(2-n)]}\,dx}{u_e^{\{[3-2\beta(1-n)][2-\beta(2-3n)]\}/[2-\beta(2-n)]}\,x} \quad (23b)$$

$$Re_x = \rho u_e^{2-n} x^n/\gamma_1. \quad (23c)$$

The substitution of equation (23a) into equations (18) and (11) yields

$$C_{fx} Re_x^{\beta/(1+\beta n)} = \frac{2\Omega}{\left\{\left[\frac{2\Omega[2-\beta(2-3n)][1-\beta(1-n)][1+\beta n]}{\beta n[2-\beta(2-n)]}\right]^{\beta n/(1+\beta n)} \times I^{\beta n/(1+\beta n)}\right\}} \quad (24a)$$

and

$$\beta_1 = -\left[\frac{2[2-\beta(2-3n)][1-\beta(1-n)][1+\beta n]}{\beta n[2-\beta(2-n)]}\right]\bar{m} I \quad (24b)$$

where

$$\bar{m} = \frac{\mathrm{dln}\, u_e}{\mathrm{dln}\, x}. \qquad (24c)$$

For the special case of $\bar{m}$ being constant we have wedge flow for which

$$u_e \propto x^{\bar{m}} \qquad (25a)$$

$$I = \left\{1 + \frac{[3-2\beta(1-n)][2-\beta(2-3n)]}{[2-\beta(2-n)]}\bar{m}\right\}^{-1}. \qquad (25b)$$

The analogy factor on the basis of equation (15) for the case of the flat plate, i.e. $\bar{m} = 0$ can be written as follows:

$$St_x = \frac{C_{fx}/2}{1+(C_{fx}/2)^{1/2}[9.24(Pr_w^{3/4}-1)]}. \qquad (26)$$

Using equation (24a) for the values of C_{fx}, the above equation (26) is plotted in Fig. 1 for selected values of n (1.0, 0.8, 0.6, 0.4) and a typical chosen Reynolds number of 10^5. At $n = 1$, the curve obtained is no different from that of Nakayama *et al.* [1] who compared it with existing analogies and found good agreement. For values of n deviating from unity, a comparison of the results plotted in Fig. 1 would be desirable. There are two equations in the literature for the turbulent boundary layer flow past a flat plate—one given by Skelland [3] and the other suggested by Irvine and Karni [9]. However, before making a comparison the relevant equations have to be modified to conform with the present definition of the various terms appearing in equation (26). Skelland [3] as well as Irvine and Karni [9] have used a different form of Prandtl number. This is first converted to the following generalized Prandtl number defined as

$$Pr_x = \frac{\gamma_1 C_p}{k}\left(\frac{u_e}{x}\right)^{n-1}. \qquad (27a)$$

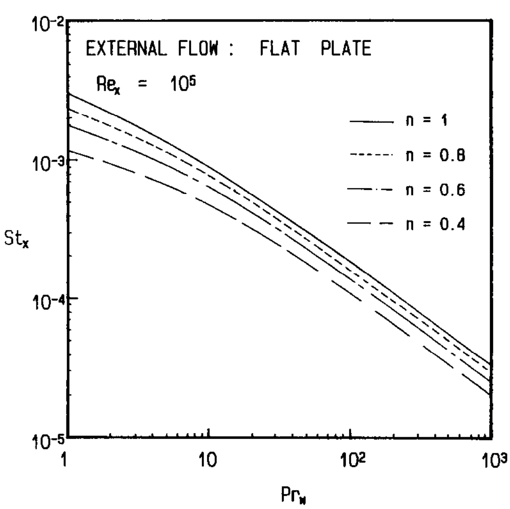

FIG. 1. Stanton number vs Prandtl number for external flow of power-law fluids past a flat plate for $\beta_1 = 0$ and $Re_x = 10^5$.

Metzner and Friend [2] have provided a relationship between the two Prandtl numbers of the form defined in equations (13c) and (27a). Their relationship which is valid for pipe flow can be easily adapted to the external to give

$$\frac{Pr_x}{Pr_w} = \left(\frac{2}{Re_x C_{fx}}\right)^{(n-1)/n}[8^{(n-1)/n}(3n+1)/4n]. \qquad (27b)$$

Using equations (27a) and (27b) and other terms conforming to the definitions used in the present analysis, the following modified forms are written:

Skelland's [3] equation

$$St_x = (0.0296)^{10/3}\left(\frac{C_{fx}}{2}\right)^{(5n+2)/3n} \times \left[8^{n-1}\left(\frac{3n+1}{4n}\right)^n Re_x\right]^{-2/3n} Pr_w^{-2/3}; \qquad (28a)$$

Irvine and Karni's [9] equation

$$St_x = \left(\frac{C_{fx}}{2}\right)\left[\left(\frac{C_{fx}}{2}\right)^{n+1} 8^{n-1} \times \left(\frac{3n+1}{4n}\right)^n Re_x\right]^{(0.4(n-1))/(n(n+1))} Pr_w^{-0.4}. \qquad (28b)$$

The results from the two equations above are plotted in Figs. 2(a) and (b). It can be seen that the present equation (26) matches with that of Skelland [3] quite closely and more so at larger values of n. However, the deviation from the equation of Irvine and Karni [9] is quite substantial. This is due to the fact that in Irvine and Karni [9] equation (28b) has $St_x \propto Pr_w^{-0.4}$ as against that of Skelland [3] equation (28a) which has $St_x \propto Pr_w^{-2/3}$ and the present analysis equation (26) which has $St_x \propto Pr_w^{-3/4}$ for high Prandtl numbers. The proportionality obtained in Skelland [3] has often been used. However, the Irvine and Karni [9] proportionality function is unknown in the literature. In fact, even for Newtonian fluids, the equation proposed by Irvine and Karni [9] does not give expected results. In the present case, the Newtonian results match very well with those of Skelland [3] especially at lower Prandtl numbers. At higher Prandtl numbers, equation (26) presented herein gives more realistic results because it shows a dependence of $St \propto Pr_w^{-3/4}$ which has been indicated by Diamant and Poreh [21] as the preferred dependence based on their own theoretical analysis supported by other analyses and confirmed by experimental data.

The effects of pressure gradient based on equation (15) are shown in Figs. 3(a)–(c) for varying Prandtl numbers. It can be seen that the pressure gradient effects diminish with increasing Prandtl number and higher pseudoplasticity. The curve in Fig. 3(a) for $n = 1$ matches that of Nakayama *et al.* [1] exactly but there is no other theoretical equation or experimental finding to confirm the trends at different values of n.

In order to reinforce confidence that the present

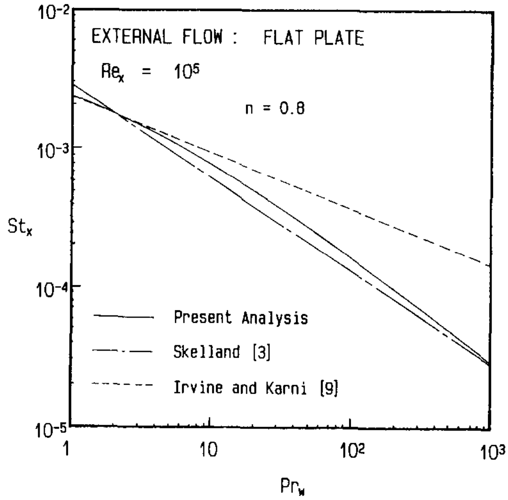

FIG. 2(a). Comparison from the predictions of the present analysis with those available in literature for external flow past a flat plate of power-law fluid with $n = 0.8$, $\beta_1 = 0$ and $Re = 10^5$.

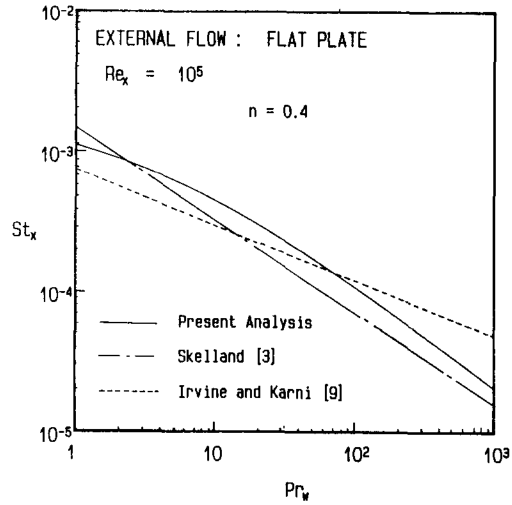

FIG. 2(b). Comparison from the predictions of the present analysis with those available in literature for external flow past a flat plate of power-law fluid with $n = 0.4$, $\beta_1 = 0$ and $Re = 10^5$.

analogy is correct for Newtonian as well as non-Newtonian power-law fluids, it is worthwhile comparing the results with other well-known and well-tested equations existing for turbulent flow of power-law fluids through smooth circular pipes. In order to do that, equation (26) needs to be adapted from the external flow case to the internal flow situation which is done as follows.

Using equations (14b) and (14c), equation (26) is rewritten as

$$\frac{2q_w}{\rho C_p u_e (T_w - T_e)} \frac{1}{2\tau_w/\rho u_e^2} = \frac{1}{1+(\tau_w/\rho u_e^2)^{1/2} P}. \quad (29)$$

Since this equation holds good at the edge of the boundary layer, it is assumed that replacing T_e by T_b (temperature of the bulk of the fluid) and replacing u_e by u_m (maximum centerline velocity for pipe flow) retains its validity. From Skelland [20], it can be seen that u_m is related to V (the average velocity) as follows:

$$u_m = (1/\psi)V \quad (30a)$$

where

$$\psi = \frac{[2-\beta(2-n)][2-\beta(2-n)]}{[1-\beta(1-n)][4-\beta(4-3n)]}. \quad (30b)$$

Thus

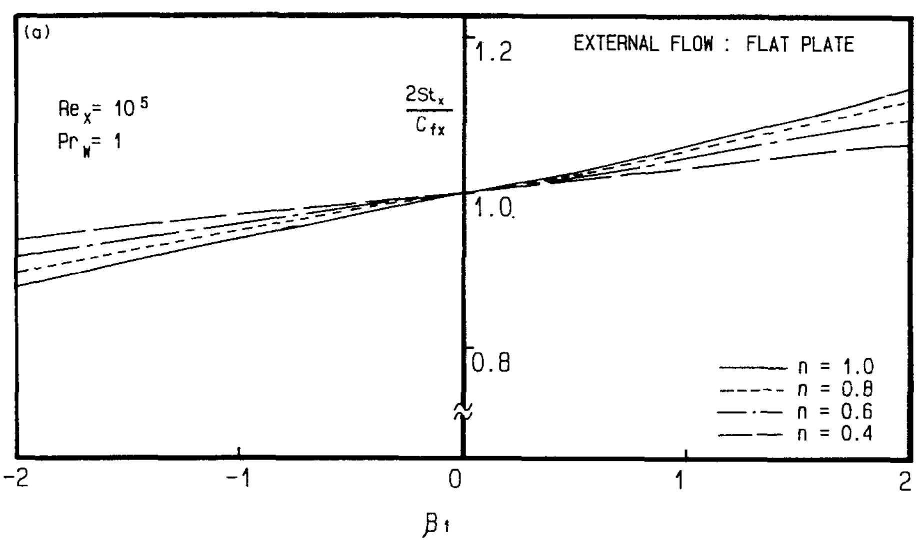

FIG. 3(a). Effects of pressure gradient on the analogy factor for external flow past a flat plate of Newtonian and power-law fluids with $Pr_w = 1$ and $Re = 10^5$.

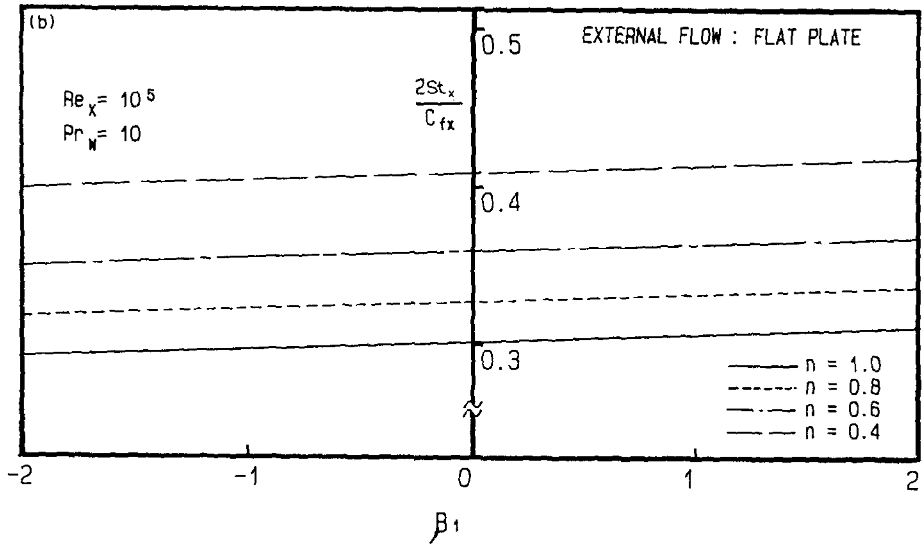

FIG. 3(b). Effects of pressure gradient on the analogy factor for external flow past a flat plate of Newtonian and power-law fluids with $Pr_w = 10$ and $Re = 10^5$.

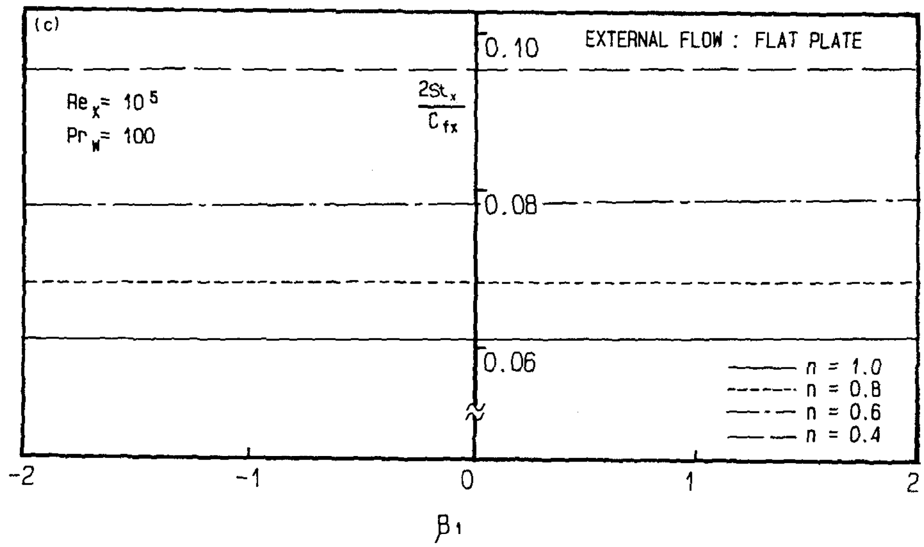

FIG. 3(c). Effects of pressure gradient on the analogy factor for external flow past a flat plate of Newtonian and power-law fluids with $Pr_w = 10^2$ and $Re = 10^5$.

$$\frac{2q_w\psi}{\rho C_p V(T_w - T_b)} \frac{1}{2\tau_w\psi/\rho V^2} = \frac{1}{1+\psi(\tau_w/\rho V^2)^{1/2}P}. \tag{31}$$

Now using the following definition:

$$f = 2\tau_w/\rho V^2 \tag{32a}$$

and the Stanton number for pipe flow

$$St = q_w/\rho C_p V(T_w - T_b) \tag{32b}$$

equation (31) can be rewritten in the simplified form using equations (32a), (32b) and (13b) as

$$St = \frac{f/2}{1/\psi + (f/2)^{1/2}[9.24(Pr_w^{3/4} - 1)]} \tag{33}$$

where the value for f is used from equation (17a) and a plot is made for varying n (1.0, 0.8, 0.6, 0.4) and Reynolds number of 10^5 as shown in Fig. 4. In order to check the propriety of equation (33), a comparison is made with the following existing theoretical expressions of Metzner and Friend [2], Sandall *et al.* [6] and Kawase and Ulbrecht [8] for two typical values of $n = 0.8$ and 0.4 as shown in Figs. 5 and 6. It should be noted that the equation proposed by Kawase and

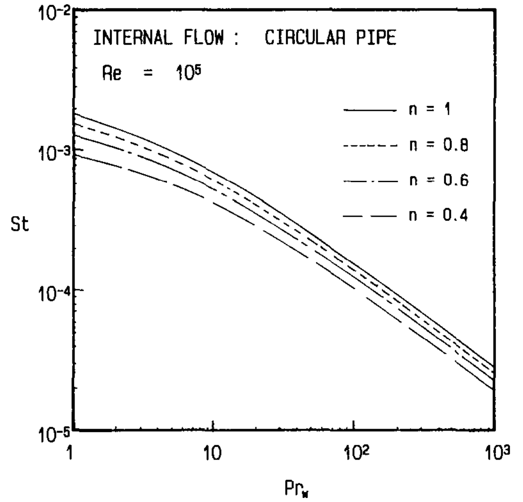

FIG. 4. Stanton number vs Prandtl number for internal flow of power-law fluids in smooth circular pipe for $\beta_1 = 0$ and $Re = 10^5$.

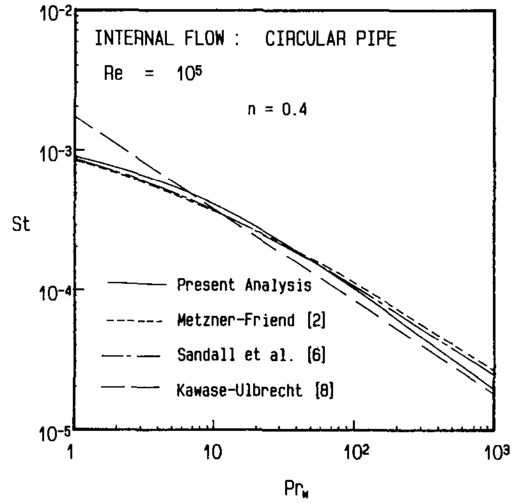

FIG. 6. Comparison from the predictions of the present analysis with those available in literature for pipe flow of power-law fluid with $n = 0.4$, $\beta_1 = 0$ and $Re = 10^5$.

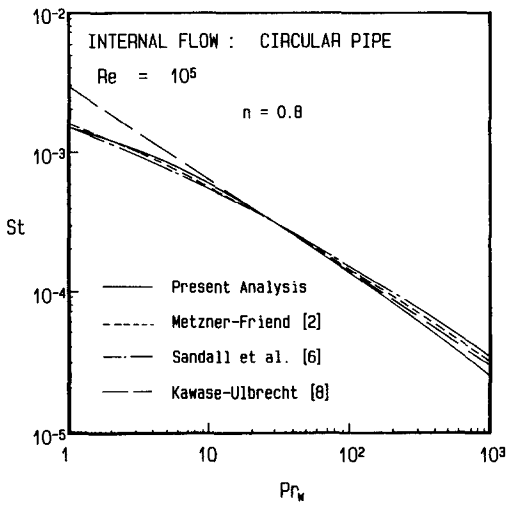

FIG. 5. Comparison from the predictions of the present analysis with those available in literature for pipe flow of power-law fluid with $n = 0.8$, $\beta_1 = 0$ and $Re = 10^5$.

Ulbrecht [8] has been appropriately modified to conform with the definitions of various terms used in the present analysis.

Metzner and Friend's [2] equation

$$St = \frac{f/2}{1.2 + 11.8(f/2)^{1/2}(Pr_w - 1)(Pr_w)^{-1/3}}, \quad (34a)$$

Sandall *et al.*'s [6] equation

$$St = \frac{(f/2)^{1/2}}{1.241 \ln Pr_w + 12.527 Pr_w^{2/3} + 2.78 \ln [Re(f/2)^{1/2}/90]^{1/n}}; \quad (34b)$$

Kawase and Ulbrecht's [8] equation

$$St = 0.075 n^{1/3} (f/2)^{1/2} Pr_w^{-2/3}. \quad (34c)$$

It can be seen from Figs. 5 and 6 that equation (33) proposed herein gives a very close match to the equations proposed by Metzner and Friend [2], Sandall *et al.* [6] and Kawase and Ulbrecht [8]. Whereas Metzner and Friend [2] have used a constant value of 1.2 for $1/\psi$ for all n, the present proposed equation uses equation (28b) to determine the said value for a different pseudoplasticity index. At $n = 1$, equation (28b) predicts $1/\psi = 1.22$. It should be noted that for all n, the present equation gives almost identical results with that of Metzner and Friend [2] and Sandall *et al.* [6] especially at lower Prandtl numbers and with Kawase and Ulbrecht [8] at higher Prandtl numbers. Whereas it is known that the model of Metzner and Friend [2] is fairly accurate up to Prandtl numbers of the order of 100, the Kawase and Ulbrecht [8] equation was developed for the high Prandtl number region. Hence, it is not surprising that the Kawase and Ulbrecht [8] equation does not give good predictions at lower Prandtl numbers. However, the present equation would provide accurate results over the entire range of Prandtl numbers from low (which is of relevance to Newtonian fluids) to high (which is of relevance to non-Newtonian power-law fluids which are known to have high consistencies). Some of the other equations available in the literature such as those presented by Krantz and Wasan [5] and Wangskarn and Ghorashi [10] are quite complex in form and require the knowledge of eddy and velocity distribution. Hence, they were not used in Figs. 5 and 6 for comparison.

A comparison of the theoretical predictions of equation (33) with experimental heat transfer data is shown in Fig. 7. The data were obtained by Raniere [11], Haines [12], Friend [13] and Farmer [14] for a very wide range of pseudoplasticity index n from 0.9 to 0.4. This data has been used earlier by Sandall *et al.* [6] as well as Kawase and Ulbrecht [8] for comparing their theoretical predictions. The present equa-

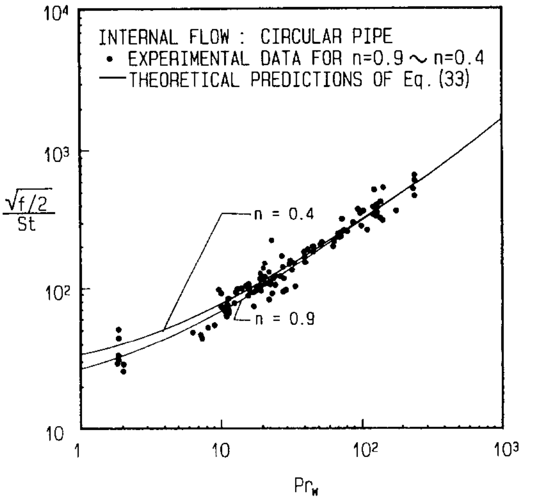

FIG. 7. Comparison from the predictions of the present analysis with experimental data available in literature for pipe flow of power-law fluids with $n = 0.9$–0.4. (Data of Raniere [11], Haines [12], Friend [13] and Farmer [14] as quoted by Sandall *et al.* [6].)

tion (33) can be seen to compare reasonably well with the experimental findings.

CONCLUSIONS

There are three equations of significance which have been presented in this work. Equation (15) is the proposed analogy for momentum/heat transfer during turbulent flow of a non-Newtonian power-law fluid past an external surface of arbitrary shape under the influence of a mild pressure gradient. Such an equation of a very general form is the first of its kind for inelastic non-Newtonian fluids. Hence, no comparison of the results could be done for this equation. The special case of simplest flow past a flat plate without a pressure gradient was therefore chosen for comparison.

Equation (26) is thus the second equation of significance and presents the momentum/heat transfer analogy for turbulent flow of a power-law fluid past a flat plate without pressure gradient. A comparison with the proposed equation of Skelland [3] shows a reasonably good agreement. The equation of Irvine and Karni [9], however, did not compare well as their proposed $St_x \propto Pr_w^{-0.4}$ is contrary to expected trends.

The third equation of significance is equation (33) which presents the momentum/heat transfer analogy for turbulent flow of a power-law fluid through smooth circular pipes. A comparison of the proposed equation with those available in the literature shows that it is more comprehensive as it predicts accurately in the low Prandtl number region where the equation of Kawase and Ulbrecht [8] fails and at the same time predicts very well in the high Prandtl number region where the equation of Metzner and Friend [2] fails. Moreover, equation (33) is very simple in its form and has no adjustable parameters, unlike some of the other equations in the literature which are complex and need the prior knowledge of the eddy and velocity distribution for evaluation.

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ANALOGIE DE TRANSFERT QUANTITE DE MOUVEMENT/CHALEUR POUR DES FLUIDES A LOI-PUISSANCE EN ECOULEMENT TURBULENT DE COUCHE LIMITE AVEC GRADIENT DE PRESSION

Résumé—Il existe plusieurs essais pour développer des analogies de transfert pour la quantité de mouvement, la chaleur et la masse dans les fluides à loi-puissance. Néanmoins aucun ne considère la présence d'un gradient de pression. Le présent texte présente une analogie entre quantité de mouvement et chaleur sous l'influence d'un gradient de pression modéré pour les fluides non newtoniens à loi-puissance, à partir de la méthodologie de Nakayama *et al.* (*AIAA J.* **22**, 841–844 (1984)) pour les fluides newtoniens.

ANALOGIE VON IMPULS- UND WÄRME-TRANSPORT IN "POWER-LAW" FLUIDEN IN EINER TURBULENTEN GRENZSCHICHTSTRÖMUNG MIT GERINGEM DRUCKGRADIENTEN

Zusammenfassung—Wie die Literatur zeigt, wurden schon zahlreiche Versuche unternommen eine Analogie für den Impuls-, Wärme- und Stofftransport in "Power-law" Fluiden zu entwickeln. Bisher wurde dabei der Druckgradient vernachlässigt. In der vorliegenden Arbeit wird der Versuch unternommen, eine Analogie zwischen Impuls- und Wärme-Transport zu entwickeln, die den Einfluß eines schwachen Druckgradienten für nicht-Newton' sche "Power-law" Fluide beinhaltet. Dazu wird das Lösungsverfahren nach Nakayama *et al.* (*AIAA J.* **22**, 841–844 (1984)) für Newton'sche Fluide verwendet.

АНАЛОГИЯ МЕЖДУ ПЕРЕНОСОМ ИМПУЛЬСА И ТЕПЛА ДЛЯ СТЕПЕННЫХ ЖИДКОСТЕЙ ПРИ ТУРБУЛЕНТНОМ ТЕЧЕНИИ В ПОГРАНИЧНОМ СЛОЕ С УМЕРЕННЫМИ ГРАДИЕНТАМИ ДАВЛЕНИЯ

Аннотация—В литературе неоднократно предпринимались попытки разработки аналогий между переносом импульса, тепла и массы к степенным жидкостям. Однако при их формулировке не учитывалось наличие градиента давления. Целью настоящего исследования является разработка аналогии между переносом импульса и тепла под влиянием умеренного градиента давления в случае неньютоновских степенных жидкостей, предложеной Накаямой и др. (*AIAA J.* **22**, 841–844 (1984)) для ньютоновских жидкостей.